Journal of Basic and Applied Engineering Research
p-ISSN: 2350-0077; e-ISSN: 2350-0255; Volume 6, Issue 5; April-June, 2019, pp. 301-303

http://www.krishisanskriti.org/Publication.html

Region Containing the Zeros of Polynomials

Irfan Ahmed Faiq

Department of Mathematics, National Institute of Technology Srinagar, India
E-mail: abliman22@yahoo.com, faiqirfan85@gmail.com

Abstract—*It was proved by Joyal, Labelle and Rahman [A. Joyal, G. Labelle and Q. I. Rahman, On the location of zeros of polynomials, Canad. Math. J., Bull., 10(1967), 53-63] that if $p > 1$, then all the zeros of $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ are contained in the circle $|z| \le k$, where $k \ge max\ (1, |a_{n-1}|)$ is a root of the equation*

$$(|z| - |a_n|)^q(|z|^q - 1) - B^q = 0, \qquad p^{-1} + q^{-1} = 1,$$

where

$$B = \left\{\sum_{j=0}^{n-2} |a_j|^p\right\}^{\frac{1}{p}}, \qquad p > 1.$$

In this paper, we not only generalize the above result but a verity of interesting results can be deduced from it.

Keywords*:* *Polynomials, Maximum Modulus, Inequalities in the complex Domain, Zeros.*
2010 Mathematics subject classification: *30A10, 30C10, 30D15.*

1. INTRODUCTION AND STATEMENT OF RESULTS

The problem of finding out the region which contains all or a prescribed number of zeros of a polynomial has long history and dates back to the earliest days when the geometric representation of complex numbers was introduced into mathematics. Since the days of Gauss [2] and Cauchy [1], many mathematicians have contributed to the further growth of the subject.

We first mention the following result due to Cauchy:

Theorem A. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If $M = max_{1\le j\le n}\left|\frac{a_j}{a_n}\right|$, then all the zeros of $P(z)$ lie in

$$|z| \le 1 + M.$$

As an improvement of Theorem A,Joyal, Labelle and Rahman [3], proved the following:

Theorem B. If $B = \left\{\sum_{j=0}^{n-2} |a_j|^p\right\}^{\frac{1}{p}},\ p > 1$ then all the zeros of $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ are contained in the circle$|z| \le k$, where $k \ge \max\ (1, |a_{n-1}|)$ is a root of the equation

$$(|z| - |a_n|)^q(|z|^q - 1) - B^q = 0, \qquad p^{-1} + q^{-1} = 1.$$

The following result is due to Montel and Marty [4, p.107].

Theorem C. All the zeros of the polynomial $P(z) = a_0 + a_1 z + \dots + a_{n-1}z^{n-1} + z^n$

Lie in $|z| \le \max\left(L,\ L^{\frac{1}{n}}\right)$, where L is the length of the polygonal line joining in the succession the points $0, a_0, a_1, \dots, a_{n-1}, 1$; that is

$$L = |a_0| + |a_1 - a_0| + \dots + |a_{n-1} - a_{n-2}| + |1 - a_{n-1}|.$$

As a generalization of Theorem A and Theorem B Mohammad [5], proved the following:

Theorem D. If $0 < a_{i-1} \le k a_i,\ k > 0$, then all the zeros of $P(z) = a_0 + a_1 z + \dots + a_n z^n$ lie in $|z| \le \max\left(M,\ M^{\frac{1}{n}}\right)$, where

$$M = \frac{(a_0 + a_1 + \dots a_{n-1})}{a_n}(k - 1) + k.$$

In this paper, we first prove a more general result which not only improves upon Theorem C and Theorem D but also a variety of interesting results can be deduced from it by a fairly uniform procedure.

Theorem 1. If $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ is a polynomial of degree n, with $Re\ a_i = \alpha_i$ and $Im\ a_i = \beta_i$ for $i = 0,1,2,\dots,n-1$, then all the zeros of $P(z)$ will lie in $|z| \le R = \max\left(L_p, L_p^{\frac{1}{n}}\right)$, where

$$L_p = (n+2)^{\frac{1}{q}}\left[\left(\sum_{i=1}^{n+2} |\alpha_{n-i+2}t_1 t_2 + \alpha_{n-i+1}(t_1 - t_2) - \alpha_{n-i}|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n+2} |\beta_{n-i+2}t_1 t_2 + \beta_{n-i+1}(t_1 - t_2) - \beta_{n-i}|^p\right)^{\frac{1}{p}}\right].$$

$a_{-1} = a_{-2} = a_{n+1} = 0,\ p^{-1} + q^{-1} = 1$and $t_1 > t_2 \ge 0$.

For $t_1 = 1$ and $t_2 = 0$, we obtain the following:

Corollary 1.1. If $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ is a polynomial of degree n, then all the zero of $P(z)$ will lie n $|z| \le R = \max\left(L_p, L_p^{\frac{1}{n}}\right)$, where

$$L_p = (n+2)^{\frac{1}{q}}\left[\left(\sum_{i=1}^{n+2}|\alpha_{n-i+1} - \alpha_{n-i}|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n+2}|\beta_{n-i+1} - \beta_{n-i}|^p\right)^{\frac{1}{p}}\right],$$
$$p^{-1} + q^{-1} = 1.$$

Taking $\beta_i = 0, \forall\, i$ and letting $q \to \infty$ in Corollary 1.1 so that $p \to 1$ and $(n+2)^{\frac{1}{q}} \to 1$, we obtain Theorem C, a result due to Montel and Marty.

Next, we prove:

Theorem 2. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with $Re\, a_i = \alpha_i$ and $Im\, a_i = \beta_i$ for $i = 0,1,2,\ldots,n,$ such that for some $t_1 > t_2 \ge 0$ with

$$a_i t_1 t_2 + a_{i-1}(t_1 - t_2) - a_{i-2} \ge 0, \quad i = 1,2,\ldots,n+1, \quad a_{-1} = a_{-2} = a_{n+1} = 0,$$

then all the zeros of $P(z)$ will lie in $|z| \le \max\left(M, M^{\frac{1}{n}}\right)$, where

$$M = \frac{(t_1 - 1)(t_2 + 1)\sum_{i=0}^{n-1}(\alpha_i + \beta_i) + (t_2(t_1 - 1) + t_1)(\alpha_n + \beta_n)}{|a_n|}.$$

For $t_2 = 0$, we obtain the following result:

Corollary 1.2. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with $Re\, a_i = \alpha_i$ and $Im\, a_i = \beta_i$ for $i = 0,1,2,\ldots,n,$ such that for some $t_1 > 0$ with

$$a_{i-1}t_1 - a_{i-2} \ge 0, \quad i = 1,2,\ldots,n+1, \quad a_{-1} = a_{-2} = a_{n+1} = 0,$$

then all the zeros of $P(z)$ will lie in $|z| \le \max\left(M, M^{\frac{1}{n}}\right)$, where

$$M = \frac{(t_1 - 1)\sum_{i=0}^{n-1}(\alpha_i + \beta_i) + t_1(\alpha_n + \beta_n)}{|a_n|}.$$

For $t_1 = k$ and $\beta_i = 0\ \forall\, i$we get Theorem D, result due to Mohammad. Further for $t_1 = 1$, we get Enestrom Kakeya Theorem.

2. PROOF OF THEOREMS

Proof of Theorem 1. Consider

$$F(z) = (t_2 + z)(t_1 - z)P(z).$$

$$= \{t_1 t_2 + (t_1 - t_2)z - z^2\}\{z^n + [\alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0] + i[\beta_{n-1}z^{n-1} + \cdots + \beta_1 z + \beta_0]\}.$$

$$= -z^{n+2} + [\{(t_1 - t_2) - \alpha_{n-1}\}z^{n+1} + \{t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}\}z^n + \cdots$$
$$+ \{\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0\}z^2 + \{\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)\}z + \alpha_0 t_1 t_2]$$
$$+ i[\{(t_1 - t_2) - \beta_{n-1}\}z^{n+1} + \{t_1 t_2 + \beta_{n-1}(t_1 - t_2) - \beta_{n-2}\}z^n + \cdots$$
$$+ \{\beta_2 t_1 t_2 + \beta_1(t_1 - t_2) - \beta_0\}z^2 + \{\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)\}z + \beta_0 t_1 t_2].$$

Therefore

$$|F(z)| \ge |z|^{n+2}\left\{1 - \left(\sum_{i=1}^{n+2}\frac{|\alpha_{n-i+2}t_1 t_2 + \alpha_{n-i+1}(t_1 - t_2) - \alpha_{n-i}|}{|z|^i}\right) + \left(\sum_{i=1}^{n+2}\frac{|\beta_{n-i+2}t_1 t_2 + \beta_{n-i+1}(t_1 - t_2) - \beta_{n-i}|}{|z|^i}\right)\right\}.$$

$$\ge |z|^{n+2}\left\{1 - (n+2)^{\frac{1}{q}}\left(\sum_{i=1}^{n+2}\frac{|\alpha_{n-i+2}t_1 t_2 + \alpha_{n-i+1}(t_1 - t_2) - \alpha_{n-i}|^p}{|z|^{ip}}\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n+2}\frac{|\beta_{n-i+2}t_1 t_2 + \beta_{n-i+1}(t_1 - t_2) - \beta_{n-i}|^p}{|z|^{ip}}\right)^{\frac{1}{p}}\right]\right\}. \quad (1.2.1)$$

If $L_p \ge 1, \max\left(L_p, L_p^{\frac{1}{n}}\right) = L_p$.Let $|z| \ge 1$, then $\frac{1}{|z|^{ip}} \le \frac{1}{|z|^p}(i = 1,2,\ldots,n+2)$. Hence inequality (1.2.1) implies that if $|z| > L_p$, then

$$|F(z)| \ge |z|^{n+2}\left\{1 - \frac{(n+2)^{\frac{1}{q}}}{|z|}\left[\left(\sum_{i=1}^{n+2}|\alpha_{n-i+2}t_1 t_2 + \alpha_{n-i+1}(t_1 - t_2) - \alpha_{n-i}|^p\right)^{\frac{1}{p}}\right.\right.$$

$$+\left(\sum_{i=1}^{n+2}|\beta_{n-i+2}t_1t_2 + \beta_{n-i+1}(t_1-t_2)-\beta_{n-i}|^p\right)^{\frac{1}{p}}\Bigg]\Bigg\}.$$

$$= |z|^{n+2}\left(1 - \frac{L_p}{|z|}\right).$$

$$> 0.$$

Again If $L_p \le 1, \max\left(L_p, L_P^{\frac{1}{n}}\right) = L_P^{\frac{1}{n}}$. Let $|z| \le 1$, then $\frac{1}{|z|^{ip}} \le \frac{1}{|z|^{np}} (i = 1, 2, \ldots, n+2)$. Hence by inequality (1.2.1), if$|z| > L_P^{\frac{1}{n}}$, then

$$|F(z)| \ge |z|^{n+2}\Bigg\{1 - \frac{(n+2)^{\frac{1}{q}}}{|z|^n}\Bigg[\left(\sum_{i=1}^{n+2}|\alpha_{n-i+2}t_1t_2 + \alpha_{n-i+1}(t_1-t_2)-\alpha_{n-i}|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n+2}|\beta_{n-i+2}t_1t_2 + \beta_{n-i+1}(t_1-t_2)-\beta_{n-i}|^p\right)^{\frac{1}{p}}\Bigg]\Bigg\}.$$

$$= |z|^{n+2}\left(1 - \frac{L_p}{|z|^n}\right).$$

$$> 0.$$

Hence $F(z)$ does not vanish for $|z| > \max\left(L_p, L_P^{\frac{1}{n}}\right)$. Therefore all zeros of $F(z)$ will lie in $|z| \le R = \max\left(L_p, L_P^{\frac{1}{n}}\right)$. Since all the zeros of $P(z)$ are also the zeros of $F(z)$ the result follows.

Proof of Theorem 2. Consider

$$F(z) = (t_2 + z)(t_1 - z)P(z).$$

$$= \{t_1t_2 + (t_1-t_2)z - z^2\}\{a_nz^n + [\alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0] + i[\beta_{n-1}z^{n-1} + \cdots + \beta_1 z + \beta_0]\}.$$

$$= -a_nz^{n+2} + [\{(t_1-t_2)-\alpha_{n-1}\}z^{n+1} + \{t_1t_2 + \alpha_{n-1}(t_1-t_2)-\alpha_{n-2}\}z^n + \cdots$$
$$+\{\alpha_2t_1t_2 + \alpha_1(t_1-t_2)-\alpha_0\}z^2 + \{\alpha_1t_1t_2+\alpha_0(t_1-t_2)\}z + \alpha_0t_1t_2]$$
$$+i[\{(t_1-t_2)-\beta_{n-1}\}z^{n+1} + \{t_1t_2 + \beta_{n-1}(t_1-t_2)-\beta_{n-2}\}z^n + \cdots$$
$$+\{\beta_2t_1t_2 + \beta_1(t_1-t_2)-\beta_0\}z^2 + \{\beta_1t_1t_2+\beta_0(t_1-t_2)\}z + \beta_0t_1t_2].$$

Applying Theorem 1 to the polynomial$\frac{F(z)}{a_n}$, we have all the zeros of $P(z)$ lie in $|z| \le \max\left(L_1, L_1^{\frac{1}{n}}\right)$,

$$L_1 = \frac{\sum_{i=0}^{n+1}|\alpha_it_1t_2 + \alpha_{i-1}(t_1-t_2)-\alpha_{i-2}|}{|a_n|} + \frac{\sum_{i=0}^{n+1}|\beta_it_1t_2 + \beta_{i-1}(t_1-t_2)-\beta_{i-2}|}{|a_n|}.$$

$$= \frac{t_1t_2(\alpha_0+\alpha_1+\ldots+\alpha_n) + (t_1-t_2)(\alpha_0+\alpha_1+\ldots+\alpha_n) - (\alpha_0+\alpha_1+\ldots+\alpha_{n-1})}{|a_n|}$$

$$+ \frac{t_1t_2(\beta_0+\beta_1+\ldots+\beta_n) + (t_1-t_2)(\beta_0+\beta_1+\ldots+\beta_n) - (\beta_0+\beta_1+\ldots+\beta_{n-1})}{|a_n|}.$$

$$= \frac{(t_1-1)(t_2+1)\sum_{i=0}^{n-1}(\alpha_i+\beta_i) + (t_2(t_1-1)+t_1)(\alpha_n+\beta_n)}{|a_n|}.$$

$$= M.$$

This completes the proof.

REFERENCES

[1] A. L. Cauchy, Exercise de mathematique. Oeuvres. (2), 9(1829), 122.

[2] K. F. Gauss, BeitrogeZurTheorie der algebraischenGleichungen, Wiss. Gottingen, 4(1850). Ges., Werke. 3, 73-102.

[3] A. Joyal, G. Labelle and Q. I. Rahman, On the location of zeros of polynomials,Canad. Math. J., Bull., 10(1967), 53-63.

[4] M. Marden, The geometry of zeros, Amer, Math. Soc. Math. Surveys, NewYork, 3(1949).

[5] Q. G. Mohammad, Location of zeros of polynomials,Amer. Math. Monthly,74, 3(1967), 290-292.